

# Comment on revised version of “The Hadamard circulant conjecture”

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## Abstract

The revised version of the claim by Hurley, Hurley and Hurley to have proved the circulant Hadamard matrix conjecture is mistaken.

In January 2011, Hurley, Hurley and Hurley [2] claimed to have proved the circulant Hadamard matrix conjecture, but the proof was mistaken [1]. In September 2011, a revised version [3] of the paper [2] was posted to the arXiv, with the comment that “This is post publication revision of on-line Bull. London Math. Soc. version which changes subsection 3.3.” We show that the revised version is also mistaken, by summarising part of the argument of [3] and then presenting a counterexample.

A *2-block* is a matrix of the form  $D = \begin{bmatrix} i & j \\ j & i \end{bmatrix}$  for  $i, j \in \{1, -1\}$ , and is *even* if  $i = j$  and *odd* if  $i = -j$ . Suppose there exists a circulant Hadamard matrix  $H$  of order  $4n$ . Reorder the rows and columns of  $H$  to form a  $2n \times 2n$  matrix  $M$  whose entries are 2-blocks, as in [3, p.7], and write the first row of  $M$  as  $[M_0 \ M_1 \ \dots \ M_{2n-1}]$ . Then exactly  $n$  of the 2-blocks  $M_i$  are even, and

$$\sum_{i : M_i \text{ and } M_{i+u} \text{ are even}} M_i M_{i+u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for each } u \neq 0, \quad (1)$$

where all matrix subscripts are reduced modulo  $2n$ . Fix  $u \neq 0$ . Then from (1), for each  $i$  such that  $M_i$  and  $M_{i+u}$  are even, we can assign a unique  $\ell$  such that  $M_\ell$  and  $M_{\ell+u}$  are even and such that  $M_\ell M_{\ell+u} = -M_i M_{i+u}$ . We then also assign  $i$  to  $\ell$ , write  $(i, i+u) \sim (\ell, \ell+u)$ , and call the index pairs  $(i, i+u)$  and  $(\ell, \ell+u)$  *matching*.

An even 2-block  $M_i$  is *symmetric* when the 2-block  $M_{i+n}$  is also even. The following argument is given [3, p.8] to claim that “every even block is symmetric” when  $n > 1$ . Suppose, for a contradiction, that  $M_i$  is an even block that is not symmetric. Since  $n > 1$ , there is an even 2-block  $M_{i+u}$  for some  $u \neq 0$ , and there must be a pair matching  $(i, i+u)$ . In each of five exhaustive cases,

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this forces the existence of a further pair of even 2-blocks  $(M_j, M_{j+v})$  for some  $j$  and  $v$ , where  $M_j$  is not symmetric, and there must be a pair matching  $(j, j+v)$ . Repeat this procedure. Since this procedure “cannot continue indefinitely,” we obtain a contradiction.

The following is a counterexample to this claimed procedure, using  $n = 3$  and only the first of the five specified cases:

$$(M_0, M_1, M_2, M_3, M_4, M_5) = \left( \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix} \right)$$

(writing  $+$  for 1 and  $-$  for  $-1$ ). The even 2-blocks are  $M_0$ ,  $M_2$ , and  $M_4$ , none of which is symmetric. Assign the matchings  $(0, 2) \sim (2, 4)$  and  $(0, 4) \sim (4, 2)$ . Let  $i = 0$  and  $j = 2$ , and follow the procedure of [3, p.8]. Since  $(0, 2) \sim (2, 4)$ , there must be a pair matching  $(0, 4)$ . Then, since  $(0, 4) \sim (4, 2)$ , there must be a pair matching  $(0, 2)$ . However  $(0, 2)$  already has a matching pair  $(2, 4)$ , so the claimed contradiction does not arise.

## References

- [1] R. Craigen and J. Jedwab. Comment on “The Hadamard circulant conjecture”. [arXiv:1111.3437v1](#) [math.CO].
- [2] B. Hurley, P. Hurley, and T. Hurley. The Hadamard circulant conjecture. *Bull. London Math. Soc.*, 2011. doi:10.1112/blms/bdq112.
- [3] B. Hurley, P. Hurley, and T. Hurley. The Hadamard circulant conjecture. [arXiv:1109.0748v1](#) [math.RA].